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Soliton equations in N-dimensions as exact reductions of Self-dual Yang - Mills equation IV. The (2+1)-dimensional mM-LXII and Bogomolny equations [1]

Kur. Myrzakul and R. Myrzakulov [2]

Centre for Nonlinear Problems, PO Box 30, 480035, Alma-Ata-35, Kazakhstan

Abstract

Some aspects of the multidimensional soliton geometry are considered. The relation between soliton equations in 2+1 dimensions and the Self-Dual Yang-Mills and Bogomolny equations are discussed.

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[2] E-mail: cnlpmyra@satsun.sci.kz

Contents

1 Introduction

1985. R. S. Ward: *"many (and perhaps all?) of the ordinary and partial differential equations that are regarded as being integrable or solvable may be obtained from the self-dual gauge field equations (or its generalizations) by reductions"* [1].

1991. M. J. Ablowitz and P. A. Clarkson: *"This strongly suggests that the KP and DS equations can be obtained from SDYM by exact reductions (i.e., no asymptotic limits)"* [1].

1996. R.M. : *"Many soliton equations in 2+1 dimensions, such as, the DS, KP, mKP, KdV, mKdV (and so on) equations are exact reductions of the M-LXII or mM-LXII*

equations. At the same time, their spin equivalent counterparts such as the Ishimori, M-X, M-I, M-IX (and so on) equations are exact reductions of the mM-LXI or M-0 equations" [20].

1997. At present, it is well known that the SDYM equation (SDYME) is integrable [2,23] and contains as reductions many soliton equations (SEs) [3-7, 23-28, 35]. Moreover from the SDYME can obtains may be the more known representatives of (2+1)-SEs -the KP and DS equations in asymptotic limit [1].

So some SEs are exact reductions of the SDYME, at the same time, some SEs are exact reductions of the "absolutely" other equations (from the soliton geometry), e.g., of the mM-LXII equation. In this note, we try understand this situation, that is, first aim of this paper.

Also we try understand some aspects of the multidimensional soliton geometry, that is, second aim of this note (on the soliton geometry see e.g. [8-20, 36-38]).

2 Equations of surfaces in 1+1 dimensions

In this section we review briefly the some known basic facts from 1+1 dimensions to set our notations and terminology. We consider a surface evolving in 3-dimensional space. We denote local coordinates of the surfaces by $u^1 = x, u^2 = t$. The surface is specified by the position vector $\mathbf{r}(u^1, u^2) = (x^1, x^2, x^3)$. Let the metric, i.e., the first fundamental form of this surface, be given by

$$I = d\mathbf{r}^2 = g_{\alpha\beta} du^\alpha u^\beta \tag{1}$$

As usual, the extrinsic curvature, i.e., the second fundamental form is defined as

$$II = -d\mathbf{r}d\mathbf{n} = b_{\alpha\beta} du^\alpha u^\beta \tag{2}$$

In (1)-(2)

$$g_{\alpha\beta} = \mathbf{r}_\alpha \mathbf{r}_\beta, \quad b_{\alpha\beta} = \mathbf{r}_{\alpha\beta} \dot{\mathbf{n}}, \quad \mathbf{r}_\alpha = \frac{\partial \mathbf{r}}{\partial u^\alpha} \tag{3}$$

$\mathbf{n} = (n^1, n^2, n^3)$ is the unit normal vector and

$$n^i = (detg)^{-\frac{1}{2}} \epsilon^{ikm} \frac{\partial x^k}{\partial u^1} \frac{\partial x^m}{\partial u^2} \tag{4}$$

Here ϵ^{ikm} is a totally antisymmetric tensor with $\epsilon^{123} = 1$. Repeated indices are summed on unless otherwise noted. The Gauss-Weingarden equation (GWE) reads

$$\mathbf{r}_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} \mathbf{r}_\gamma + b_{\alpha\beta} \mathbf{n} \tag{5a}$$

$$\mathbf{n}_\alpha = -g^{\gamma\beta} b_{\alpha\gamma} \mathbf{r}_\beta \tag{5b}$$

Here, the Christoffel symbols $\Gamma^\gamma_{\alpha\beta}$ of the second kind defined as usual

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\lambda} \left(\frac{\partial g^{\lambda\beta}}{\partial u^\alpha} + \frac{\partial g^{\alpha\lambda}}{\partial u^\beta} - \frac{\partial g^{\alpha\beta}}{\partial u^\lambda} \right) \tag{6}$$

From the compatibility conditions of (5), we get the Gauss-Codazzi-Mainardi-Peterson equation (GCMPE)

$$R_{\gamma\alpha\beta\lambda} = b_{\gamma\beta} b_{\alpha\lambda} - b_{\gamma\lambda} b_{\alpha\beta} \tag{7a}$$

$$D_\mu b_{\alpha\beta} = D_\alpha b_{\mu\beta} \tag{7b}$$

In the above,

$$R^\gamma_{\alpha\beta\lambda} = \frac{\partial \Gamma^\gamma_{\alpha\lambda}}{\partial u^\beta} - \frac{\partial \Gamma^\gamma_{\alpha\beta}}{\partial u^\lambda} + \Gamma^\gamma_{\mu\beta} \Gamma^\mu_{\alpha\lambda} - \Gamma^\gamma_{\mu\lambda} \Gamma^\mu_{\alpha\beta} \tag{8}$$

is the Riemann tensor and

$$D_\mu f_\alpha = \frac{\partial}{\partial u^\mu} f_\alpha - \Gamma^\gamma_{\alpha\mu} f_\gamma, \quad D_\mu f^\alpha = \frac{\partial}{\partial u^\mu} f^\alpha + \Gamma^\alpha_{\lambda\mu} f^\lambda \tag{9}$$

are the covariant derivatives. The Gaussian curvature K and the mean curvature H of the surface are

$$K = det(g^{\mu\nu} b_{\nu\lambda}) = \frac{1}{g} R_{1212}, \quad g = det(g_{ij}) \tag{10}$$

$$H = tr(g^{\mu\nu} b_{\nu\lambda}) = \frac{1}{2} g^{\mu\nu} b_{\mu\nu} \tag{11}$$

Among the global characteristics of surfaces we mention the integral curvature

$$\chi = \frac{1}{2\pi} \int_S K g^{\frac{1}{2}} d^2u \tag{12}$$

where K is the Gaussian curvature and integration in (12) is performed over the surface. For compact oriented surfaces

$$\chi = 2(1-q) \tag{13}$$

where q is the genus of the surface and we will generally assume that surfaces are compact and oriented unless otherwise specified.

Sometimes, it is convenient to work in orthogonal basis

$$\mathbf{e}_1 = \frac{1}{g_{11}} \mathbf{r}_{u^1}, \quad \mathbf{e}_2 = \mathbf{n}, \quad \mathbf{e}_3 = \mathbf{e}_1 \wedge \mathbf{e}_2 \tag{14}$$

Then the GWE takes the form

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = C \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{15a}$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = G \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{15b}$$

where

$$C = \begin{pmatrix} 0 & k & -\sigma \\ -\beta k & 0 & \tau \\ \beta\sigma & -\tau & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\beta\omega_3 & 0 & \omega_1 \\ \beta\omega_2 & -\omega_1 & 0 \end{pmatrix} \tag{16}$$

and k, τ, σ are some functions of $g_{\alpha\beta}, b_{\alpha\beta}$. Here $\mathbf{e}_j$ is the moving trihedral of a surface and

$$\mathbf{e}_1^2 = \beta = \pm 1, \quad \mathbf{e}_2^2 = \mathbf{e}_3^2 = 1 \tag{17}$$

k, σ and τ are called the normal curvature, geodesic curvature, and geodesic torsion, respectevily. So GCMPE takes the form

$$C_t - G_x + [C, G] = 0 \tag{18}$$

We note that for the curves case, the equation (15a) is as $\sigma = 0$ the Serret - Frenet equation (SFE). Let us rewrite the equation (18) in the form

$$U_t - V_x + [U, V] = 0 \tag{19}$$

where

$$U = \frac{1}{2i}\begin{pmatrix} \tau & k - i\sigma \\ k + i\sigma & -\tau \end{pmatrix}, \quad V = \frac{1}{2i}\begin{pmatrix} \omega_1 & \omega_3 - i\omega_2 \\ \omega_3 + i\omega_2 & -\omega_1 \end{pmatrix} \tag{20}$$

The Lax representation (LR) of the equation (18)-(19) is given by

$$\psi_x = U\psi \tag{21a}$$

$$\psi_t = V\psi \tag{21b}$$

Now let us give the interpretation of the above considered equations from the point of view of our formalism. In our formalism, usually, we assume that [8]

$$k = \sum_j \lambda^j k_j, \quad \tau = \sum_j \lambda^j \tau_j, \quad \sigma = \sum_j \lambda^j \sigma_j, \quad \omega_k = \sum_j \lambda^j \omega_{kj} \tag{22a}$$

where $j = 0, \pm 1, \pm 2, \pm 3, ...$. Sometimes, instead of (22a) we take the following more general case

$$k = \sum_j h_{1j} k_j, \quad \tau = \sum_j h_{2j} \tau_j, \quad \sigma = \sum_j h_{3j} \sigma_j, \quad \omega_k = \sum_j h_{4j} \omega_{kj} \tag{22b}$$

Here $h_{ij} = h_{ij}(\lambda)$ are some functions of λ , where λ is some characteristic parameter of curves or surfaces or some function of such parameters. In the cases (22), i.e. when $k, \tau, \sigma, \omega_j$ are some functions of λ, the equations (15), (18)=(19) and (21), we call the (1+1)-dimensional mM-LXI, mM-LXII

and M-LXIX equations respectively. [These conditional notations we use in order to accurately distinguish these equations from the case when $k, \tau, \sigma, \omega_j$ are independent of λ and for convenience in our internal working kitchen]. Particular cases.

i) *The Gauss-Weingarden and Gauss-Codazzi-Mainardi-Peterson equations.* These equations correspond to the case

$$k = k_0, \quad \tau = \tau_0, \quad \sigma = \sigma_0, \quad \omega_k = \omega_{k0} \tag{23}$$

i.e., when k, τ, σ are independent of λ.

ii) *The ZS-AKNS problem.* The famous Zakharov- Shabat-Ablowitz-Kaup-Newell-Segur (ZS-AKNS) spectral problem which generate many soliton equations in 1+1 is the particular case of the M-LXIX equation (21a) as

$$k_0 = i(p+q), \quad k_j = 0, \quad j \neq 0 \tag{24a}$$

$$\sigma_0 = p - q, \quad \sigma_j = 0, \quad j \neq 0 \tag{24b}$$

$$\tau_1 = -2, \quad \tau_j = 0, \quad j \neq 1 \tag{24c}$$

iii) *The Kaup-Newell-Wadati-Konno-Ichicawa spectral problem.* This case corresponds to the reduction

$$k_1 = i(p+q), \quad k_j = 0, \quad j \neq 1 \tag{25a}$$

$$\sigma_1 = p - q, \quad \sigma_j = 0, \quad j \neq 1 \tag{25b}$$

$$\tau_1 = -2, \quad \tau_j = 0, \quad j \neq 1 \tag{25c}$$

iv) *The equations of a principal chiral field.* The equations of a principal chiral field for functions u, v

$$u_t + \frac{1}{2}[u, v] = 0, \quad v_x - \frac{1}{2}[u, v] = 0 \tag{26}$$

can equally be represented with the aid of (22b), if we choose in this case

$$U = \frac{u}{1-\lambda}, \quad V = \frac{v}{\lambda+1} \tag{27}$$

v) *The (1+1)-dimensional mM-LXVI equation.* This case corresponds to the reduction when

$$\tau^2 \pm k^2 \pm \sigma^2 = n^2 \tag{28}$$

The mM-LXVI equation contains in particular spin systems. In fact, let

$$k = nS_1, \quad \sigma = nS_2, \quad \tau = nS_3 \tag{29}$$

Then we get

$$S_3^2 \pm S_1^2 \pm S_2^2 = 1 \tag{30}$$

So that in this case the M-LXIX equation (21) is LR of the (1+1)-dimensional M-O equation

$$S_t - \frac{1}{n} V_x + [S, V] = 0 \tag{31}$$

where

$$S = \begin{pmatrix} S_3 & S^- \\ S^+ & -S_3 \end{pmatrix}, \quad S^{\pm} = S_1 \pm iS_2 \tag{32}$$

So we see that practically all (1+1)-dimensional soliton equations can be obtained from the mM-LXII equation (18)=(19) as some reductions.

3 SDYM equation

It is standard to define a Yang-Mills vector bundle over a four-dimensional manifold M with connection one-form $A = A_\mu(x^\nu)dx^\mu$. The SDYM equations in this manifolds let us write to set our notation:

$$F = *F \tag{33}$$

where F is a curvature 2-form pulled back to M from the gauge bundle $P(M, g)$, explicitly:

$$F = d\omega + \omega \wedge \omega \tag{34}$$

Here the connection 1-form ω on P takes values in the Lie algebra g of the gauge group G. In terms of Cartesian coordinates x^μ, they can be expressed as

$$F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\kappa\sigma}F_{\kappa\sigma} \tag{35}$$

where $\mu, \nu, ... = 1, 2, 3, 4,$ $\epsilon_{\mu\nu\kappa\sigma}$ stands for the completely antisymmetric tensor in four dimensions with the convention: $\epsilon_{1234} = 1$. The components of the field strength $(F_{\mu\nu})$ are given by

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{36}$$

where

$$D_\mu = \partial_\mu - A_\mu$$

are covariant derivatives. We note that the SDYM equations (38) are invariant under the gauge transformation

$$A_\mu \to \phi^{-1}A_\mu\phi - \phi^{-1}\phi_\mu \tag{37}$$

Let us introduce the standard null coordinates in Euclidean space

$$x_\alpha = t + iz, \quad x_{\bar{\alpha}} = t - iz, \quad x_\beta = x + iy, \quad x_{\bar{\beta}} = x - iy$$

In this coordinates the metric is given by

$$ds^2 = dx_\alpha dx_{\bar{\alpha}} + dx_\beta dx_{\bar{\beta}}$$

Furthermore we have

$$A_\alpha = A_t + iA_z, \quad A_{\bar{\alpha}} = A_t - iA_z, \quad A_\beta = A_x + iA_y, \quad A_{\bar{\beta}} = A_x - iA_y$$

In this notations the SDYM equation are given by

$$F_{\alpha\beta} = 0 \tag{38a}$$

$$F_{\bar{\alpha}\bar{\beta}} = 0 \tag{38b}$$

$$F_{\alpha\bar{\alpha}} - F_{\beta\bar{\beta}} = 0 \tag{38c}$$

or

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha - [A_\alpha, A_\beta] = 0 \tag{39a}$$

$$\partial_{\bar{\alpha}}A_{\bar{\beta}} - \partial_{\bar{\beta}}A_{\bar{\alpha}} - [A_{\bar{\alpha}}, A_{\bar{\beta}}] = 0 \tag{39b}$$

$$\partial_{\alpha}A_{\bar{\alpha}} - \partial_{\bar{\alpha}}A_{\alpha} - [A_{\alpha}, A_{\bar{\alpha}}] = \partial_{\beta}A_{\bar{\beta}} - \partial_{\bar{\beta}}A_{\beta} - [A_{\beta}, A_{\bar{\beta}}] \tag{39c}$$

For the equations (38)=(39), the LR has the form

$$L\Phi = 0, \quad M\Phi = 0 \tag{40}$$

where

$$L = D_{\alpha} + \lambda D_{\bar{\beta}}, \quad M = D_{\beta} - \lambda D_{\bar{\alpha}} \tag{41}$$

and λ is the spectral parameter.

4 The (2+1)-dimensional mM-LXI and mM-LXII equations in space case

In this section, we present some information from the (2+1)-dimensional soliton geometry. One of the extension of the (1+1)-dimensional M-LXI equation (15) is the following (2+1)-dimensional mM-LXI equation

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = A \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{42a}$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_y = B \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{42b}$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = D \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{42c}$$

where

$$A = \begin{pmatrix} 0 & k & -\sigma \\ -\beta k & 0 & \tau \\ \beta\sigma & -\tau & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & m_3 & -m_2 \\ -\beta m_3 & 0 & m_1 \\ \beta m_2 & -m_1 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\beta\omega_3 & 0 & \omega_1 \\ \beta\omega_2 & -\omega_1 & 0 \end{pmatrix}. \tag{43}$$

Hence, we obtain the following (2+1)-dimensional mM-LXII equation [8]

$$A_y - B_x + [A, B] = 0 \tag{44a}$$

$$A_t - D_x + [A, D] = 0 \tag{44b}$$

$$B_t - D_y + [B, D] = 0 \tag{44c}$$

This equation admits the following LR

$$g_x = Ug, \quad g_y = Vg, \quad g_t = Tg \tag{45}$$

with

$$U = \frac{1}{2i}\begin{pmatrix} \tau & k - i\sigma \\ k + i\sigma & -\tau \end{pmatrix}, \quad T = \frac{1}{2i}\begin{pmatrix} \omega_1 & \omega_3 - i\omega_2 \\ \omega_3 + i\omega_2 & -\omega_1 \end{pmatrix}$$

$$V = \frac{1}{2i}\begin{pmatrix} m_1 & m_3 - im_2 \\ m_3 + im_2 & -m_1 \end{pmatrix} \tag{46}$$

So that the mM-LXII equation (44) we can rewrite in the following form

$$F_{xy} = U_y - V_x + [U, V] = 0 \tag{47a}$$

$$F_{xt} = U_t - T_x + [U, T] = 0 \tag{47b}$$

$$F_{yt} = V_t - T_y + [V, T] = 0 \tag{47c}$$

We note that the LR (45) we can write in the more usual form

$$Lg = 0, \quad Mg = 0 \tag{48}$$

where, for example, L, M we can take in the form

$$L = \partial_x + a\lambda\partial_y - (U + a\lambda V), \tag{49a}$$

$$M = \partial_t + e\lambda^2\partial_y - (T + e\lambda^2 V) \tag{49b}$$

or

$$L = D_x + a\lambda D_y, \quad M = D_t + e\lambda^2 D_y. \tag{50}$$

Some (2+1)-dimensional soliton equations such as: DS, Zakharov, (2+1)-complex mKdV, (2+1)-dNLS and so on,

are exact reductions of the mM-LXII equation (44)=(47). As example, in next sections we show how obtain the well known representatives of (2+1)-SEs - the DS and KP equations and their spin counterparts from the mM-LXI and mM-LXII equations.

4.1 Soliton equations in 2+1 dimensions as exact reductions of the mM-LXII equation

In this section, we obtain the IE and DS equation from the mM-LXI and mM-LXII equations as some exact reductions. The IE reads as [29]

$$\mathbf{S}_t = \mathbf{S} \wedge (\mathbf{S}_{xx} + \alpha^2 \mathbf{S}_{yy}) + u_x \mathbf{S}_y + u_y \mathbf{S}_x \tag{51a}$$

$$u_{xx} - \alpha^2 u_{yy} = -2\alpha^2 \mathbf{S} \cdot (\mathbf{S}_x \wedge \mathbf{S}_y). \tag{51b}$$

We take the following identification

$$\mathbf{S} = \mathbf{e}_1 \tag{52}$$

In this case we have

$$m_1 = \partial_x^{-1}[\tau_y - \frac{\beta}{2\alpha^2} M_2^{Ish} u] \tag{53a}$$

$$m_2 = -\frac{1}{2\alpha^2 k} M_2^{Ish} u \tag{53b}$$

$$m_3 = \partial_x^{-1}[k_y + \frac{\tau}{2\alpha^2 k} M_2^{Ish} u] \tag{53c}$$

$$M_2^{IE} u = u_{xx} - \alpha^2 u_{yy} \tag{54}$$

and

$$\omega_1 = \frac{1}{k}[-\omega_{2x} + \tau\omega_3] \tag{55a}$$

$$\omega_2 = -k_x - \alpha^2 (m_{3y} + m_2 m_1) + i m_2 u_x \tag{55b}$$

$$\omega_3 = -k\tau + \alpha^2 (m_{2y} - m_3 m_1) + i k u_y + i m_3 u_x. \tag{55c}$$

Now let us introduce two complex functions q, p, according to the formulae

$$q = a_1 e^{ib_1}, \quad p = a_2 e^{ib_2} \tag{56}$$

Let a_j, b_j have the forms

$$a_1^2 = \frac{1}{4}k^2 + \frac{|\alpha|^2}{4}(m_3^2 + m_2^2) - \frac{1}{2}\alpha_R k m_3 - \frac{1}{2}\alpha_I k m_2 \qquad (57a)$$

$$b_1 = \partial_x^{-1}\{-\frac{\gamma_1}{2ia_1^2} - (\bar{A} - A + D - \bar{D})\} \qquad (57b)$$

$$a_2^2 = \frac{1}{4}k^2 + \frac{|\alpha|^2}{4}(m_3^2 + m_2^2) + \frac{1}{2}\alpha_R k m_3 - \frac{1}{2}\alpha_I k m_2 \qquad (57c)$$

$$b_2 = \partial_x^{-1}\{-\frac{\gamma_2}{2ia_2^2} - (A - \bar{A} + \bar{D} - D)\} \qquad (57d)$$

where

$$\gamma_1 = i\{\frac{1}{2}k^2\tau + \frac{|\alpha|^2}{2}(m_3 k m_1 + m_2 k_y) -$$
$$\frac{1}{2}\alpha_R(k^2 m_1 + m_3 k\tau + m_2 k_x) + \frac{1}{2}\alpha_I[k(2k_y - m_{3x}) - k_x m_3]\}. \qquad (58a)$$

$$\gamma_2 = -i\{\frac{1}{2}k^2\tau + \frac{|\alpha|^2}{2}(m_3 k m_1 + m_2 k_y) +$$
$$\frac{1}{2}\alpha_R(k^2 m_1 + m_3 k\tau + m_2 k_x) + \frac{1}{2}\alpha_I[k(2k_y - m_{3x}) - k_x m_3]\}. \qquad (58b)$$

Here $\alpha = \alpha_R + i\alpha_I$. In this case, q, p satisfy the following DS equation

$$iq_t + q_{xx} + \alpha^2 q_{yy} + vq = 0 \qquad (59a)$$

$$-ip_t + p_{xx} + \alpha^2 p_{yy} + vp = 0 \qquad (59b)$$

$$v_{xx} - \alpha^2 v_{yy} + 2[(pq)_{xx} + \alpha^2 (pq)_{yy}] = 0. \qquad (59c)$$

It is means that the IE (51) and the DS equation (59) are L-equivalent [9] to each other. As well known that these equations are G-equivalent [22] to each other [30]. A few comments are in order.

i) From these results, we get the Ishimori I and DS-I equations as $\alpha_R = 1, \alpha_I = 0$

ii) and the Ishimori II and DS-II equations as $\alpha_R = 0, \alpha_I = 1$.

iii) For DS-II equation we have

$$pq = \mid q \mid^2 = \mid p \mid^2 \tag{60}$$

iv) at the same time, for the DS-I equation we obtain

$$pq \neq \mid q \mid^2 \neq \mid p \mid^2 \tag{61}$$

$$\mid q \mid^2 = \mid p \mid^2 - km_3 \tag{62}$$

$$pq = (pq)_R + i(pq)_I \tag{63}$$

so that pq is the complex quantity.

5 The mM-LXI and mM-LXII equations in the plane case

The (2+1)-dimensional mM-LXI equation in plane has the form [8]

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}_x = A_p \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}_y = B_p \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}_t = D_p \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} \tag{64}$$

where

$$A_p = \begin{pmatrix} 0 & k \\ -\beta k & 0 \end{pmatrix}, \quad B_p = \begin{pmatrix} 0 & m_3 \\ -\beta m_3 & 0 \end{pmatrix}$$

$$D_p = \begin{pmatrix} 0 & \omega_3 \\ -\beta\omega_3 & 0 \end{pmatrix}. \tag{65}$$

In the plane case the mM-LXII equation takes the following simple form

$$k_y = m_{3x} \tag{66a}$$

$$k_t = \omega_{3x} \tag{66b}$$

$$m_{3t} = \omega_{3y} \tag{66c}$$

Hence we get

$$m_3 = \partial_x^{-1} k_y \tag{67}$$

The nonlinear evolution equation has the form (66b). Many (2+1)-dimensional integrable equations such as the Kadomtsev-Petviashvili, Novikov-Veselov (NV), mNV, KNV, (2+1)-KdV, mKdV equations are the integrable reductions of the M-LXII equation (66).

5.1 The KP and mKP equations as exact reductions of the mM-LXII equation

For example, let us show that the KP and mKP equations are exact reductions of the mM-LXII equation (66). Consider the M-X equation [8]

$$\mathbf{S}_t = \frac{\omega_3}{k}\mathbf{S}_x \tag{68}$$

where

$$\omega_3 = -k_{xx} - 3k^2 - -3\alpha^2\partial_x^{-1} m_{3y} \tag{69}$$

If we put $\mathbf{S} = \mathbf{e}_1$ then from (66) we obtain the L-equivalent counterpart of the M-X equation which is the KP equation

$$k_t + 6kk_x + k_{xxx} + 3\alpha^2 m_{3y} = 0 \tag{70a}$$

$$m_{3x} = k_y \tag{70b}$$

As known the LR of this equation is given by

$$\alpha\psi_y + \psi_{xx} + k\psi = 0 \tag{71a}$$

$$\psi_t + 4\psi_{xxx} + 6k\psi_x + 3(k_x - \alpha m_3)\psi = 0 \tag{71b}$$

6 Linear problems from the mM-LXII equation

The idea is the following. Let us rewrite the mM-LXII equation (44) in the form

$$LA = B_x \tag{72a}$$

$$MA = D_x \tag{72b}$$

where

$$L = \partial_y - [,B], \quad M = \partial_t - [,D] \tag{73}$$

For the case (47) these equations take the forms

$$LU = V_x \tag{74a}$$

$$MU = T_x \tag{74b}$$

The compatibility condition of the equations (72) and (74) are the equations (44c) and (47c), respectively. So that the equations (72) and (74) play the role of the LR for the evolution equations (44c) and (47c), respectively. In this way, we can obtain new SEs. Example. For simplicity, we consider the M-LXII equation (66). The compatibility condition of the equations (66a,b) is the equation (66c). Let we choose

$$m_3 = -\frac{1}{\alpha}(k_x + k^2 + u), \quad \omega_3 = -4k_{xx} - 12kk_x - 4k^3 - 6uk - 3u_x + 3\alpha v \tag{75}$$

Then the compatibility condition of (66a,b) gives

$$u_t + 6uu_x + u_{xxx} + 3\alpha^2 v_y = 0 \tag{76a}$$

$$v_x = u_y \tag{76b}$$

which is nothing but the KP equation (70). On the other hand, eliminating u out of equations (66a,b) with (75), one obtains the modified KP (mKP) equation

$$k_t = 6k^2 k_x - k_{xxx} + 3\alpha(2k_x w - \alpha w_y) \tag{77a}$$

$$w_x = k_y \tag{77b}$$

It is interesting to note that if define k by

$$k = \frac{\psi_x}{\psi} \tag{78}$$

then ψ satisfies the equations (71). Finally we note that the (2+1)-dimensional M-LXI and M-LXII equations are the particular cases of the (2+1)-dimensional mM-LXI and mM-LXII equations as $\sigma = 0$ respectively.

7 Soliton equations in 2+1 as reductions of the SDYM equation

As in introduction mentioned the SDYM equation contains the (1+1)-dimensional SEs as particular reductions. In this section we show that SEs in 2+1 dimensions also are exact reductions of the SDYM equation (39). For this purpose, consider the coordinates

$$x_\alpha = it, \quad x_{\bar{\alpha}} = -it, \quad x_\beta = x + iy, \quad x_{\bar{\beta}} = x - iy \tag{79}$$

Now in the SDYM equation (38)=(39) we take

$$A_\alpha = -iD, \quad A_{\bar{\alpha}} = iD, \quad A_\beta = A - iB, \quad A_{\bar{\beta}} = A + iB. \tag{80}$$

where we mention that A, B, C, D are in our case real matrices. Then the SDYM equation (39) reduces to the (2+1)-dimensional mM-LXII equation (44). So the (2+1)-dimensional mM-LXII equation is the integrable reduction of the SDYM equation.

As many (may be all) soliton equations in 2+1 dimensions are some integrable reductions of the mM-LXII and/or M-LXII equations (44) and/or (47) then as follows from the results of the previous sections these (2+1)-dimensional soliton equations are exact reductions of the SDYM equation.

8 The Bogomolny equation and SEs

It is well known that if in the SDYM equation (39) we take

$$A_\alpha = \Psi - iD, \quad A_{\bar{\alpha}} = \Psi + iD, \quad A_\beta = A - iB, \quad A_{\bar{\beta}} = A + iB. \tag{81}$$

and assume that A, B, D, Ψ are independent of z, then we obtain the following equation

$$\Psi_t + [\Psi, D] + A_y - B_x + [A, B] = 0 \tag{82a}$$

$$\Psi_y + [\Psi, B] + D_x - A_t + [D, A] = 0 \tag{82b}$$

$$\Psi_x + [\Psi, A] + B_t - D_y + [B, D] = 0 \qquad (82c)$$

which is nothing but the Bogomolny equation (BE) in Euclidean coordinates, which as known is relevant in the study of magnetic monopoles [31-34]. If in the BE (82) we put $\Psi = 0$, then we get the mM-LXII equation (44). So we have also shown that the mM-LXII equation and hence SEs in 2+1 dimensions are the particular exact reductions of the BE.

9 The M-LXVI equation and spin systems

In this section we show how spin systems can be included into our formalism. For this purpose, we consider the M-LXVI equation which is the particular case of some above considered equations as

$$\tau^2 \pm k^2 \pm \sigma^2 = n^2(x, y, t) \qquad (83)$$

Let

$$k = nS_1, \quad \sigma = nS_2, \quad \tau = nS_3 \qquad (84)$$

Then from (83) follows that

$$S_3^2 \pm S_1^2 \pm S_2^2 = 1 \qquad (85)$$

Below we consider the case when $\tau^2 + k^2 + \sigma^2 = n^2 =$ *constant*. Let us we deduce the (2+1)-dimensional M-LXVI equation. For this purpose, we consider the gauge transformation

$$\begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix} = E \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \qquad (86)$$

where $\mathbf{e}_j$ are the solutions of the mM-LXI equation (42). Then hence and from (42) we get

$$\begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix}_x = A' \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix} \qquad (87a)$$

$$\begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix}_y = B' \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix} \tag{87b}$$

$$\begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix}_t = D' \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix} \tag{87c}$$

Here

$$A' = EAE^{-1}+E_xE^{-1}, \quad B' = EBE^{-1}+E_xE^{-1}, \quad D' = EDE^{-1}+E_xE^{-1} \tag{88}$$

We can choose the function E so that

$$A' = a \begin{pmatrix} 0 & S_1 & -S_2 \\ -\beta S_1 & 0 & S_3 \\ \beta S_2 & -S_3 & 0 \end{pmatrix} \tag{89}$$

Then the compatibility condition of the equations (87) gives the (2+1)-dimensional M-0 equation

$$A'_y - B'_x + [A', B'] = 0 \tag{90a}$$

$$A'_t - D'_x + [A', D'] = 0 \tag{90b}$$

$$B'_t - D'_y + [B', D'] = 0 \tag{90c}$$

The LR of this equation has the form

$$\psi_x = aS\psi \tag{91a}$$

$$\psi_y = V'\psi \tag{91b}$$

$$\psi_t = T'\psi \tag{91c}$$

So that the (2+1)-dimensional M-0 equation take the form

$$S_t - \frac{1}{a}V'_x + [S, V'] = 0 \tag{92a}$$

$$S_y - \frac{1}{a}T'_x + [S, T'] = 0 \tag{92b}$$

$$V'_t - T'_y + [V', T'] = 0 \tag{92c}$$

Finally we note that the M-0 equation (90)=(92) admits many integrable spin systems in 2+1 dimensions.

10 Summary

So, we have considered some aspects of the multidimensional soliton geometry. Our approach permits find some integrable classess of curves and surfaces in multidimensions. Also we have shown that the mM-LXII equation is exact reduction of the Bogomolny hence and SDYM equations. As many soliton equations , for example, in 2+1 dimensions are particular cases of the mM-LXII (and/or M-LXII) equation, it is means that they are in turn exact reductions of the SDYM equation and/or the Bogomolny equation. The connection between spin systems and curves/surfaces is also discussed. It is shown that the M-0 equation which generate spin systems also is exact reduction of the Bogomolny and SDYM equations. So the Ward's conjecture (see Introduction) is justified (once more) and in 2+1 dimensions. However, many questions remain open and deserve further investigation. So the further studies of these questions seem to be very interesting.

A few words on soliton geometry in 3+1 dimensions. In 3+1 dimensions, the equation (15) admits several extensions. Some of them are as follows:

i) The (3+1)-dimensional mM-LXI equation [8]

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}_x = A \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}_y = B \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}_z = C \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}_t = D \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix} \qquad (93)$$

ii) The (3+1)-dimensional mM-LXVIII equation [8]

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}_{x_\alpha} = -\lambda \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}_{x_{\bar{\beta}}} + A \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix} \tag{94a}$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}_{x_\beta} = \lambda \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}_{x_{\bar{\alpha}}} + B \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix} \tag{94b}$$

and so on.

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12 Questions

Guestion-1: Please, give the twistor description of the above presented results.
Question-2: Please, consider the connection between the above results and the Self-Dual Einstein equation and hyper-Kahler hierarchies.

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